ELEN 4810 Homework 3

Analytical Questions

4.25 The Fourier transform $W(j\Omega)$ is the convolution of the Fourier transforms of x_1 and x_2. It is supported within the interval

$$-\Omega_1 - \Omega_2 \le \Omega \le \Omega_1 + \Omega_2. \tag{1}$$

Hence, w is bandlimited, with bandlimit $\Omega_M = \Omega_1 + \Omega_2$. The largest feasible sampling period is (corresponding to Nyquist rate $\Omega_s = 2\Omega_M$) is

$$T = \frac{2\pi}{\Omega_s} = \frac{\pi}{\Omega_1 + \Omega_2}. \tag{2}$$

4.27 Because we sample at twice the bandlimit, the composite system is LTI when restricted to bandlimited inputs.

(a). $H_c(j\Omega) = H_d(e^{j\Omega T}) = T^{-1}(e^{j\Omega T/2} - e^{-j\Omega T/2})$.

(b). Here, $x_d[n] = \frac{\sin(\Omega_M nT)}{\Omega_M nT} = \frac{\sin(n\pi)}{n\pi}$, and thus

$$x_d[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{else} \end{cases} \tag{3}$$

Using the overall frequency response of the system, $Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega) = T^{-1}(e^{j\Omega T/2}X_c(j\Omega) - T^{-1}e^{-j\Omega T/2}X_c(j\Omega)$. Thus

$$y_c(t) = T^{-1}x_c(t + T/2) - T^{-1}x_c(t - T/2) \tag{4}$$

is a difference of sinc functions. Finally, because $y_c(t)$ is an ideal bandlimited interpolation of $y_d[n]$, we have

$$y_d[n] = y_c(nT) = T^{-1}x_c(nT + T/2) - T^{-1}x_c(nT - T/2) \tag{5}$$

We can produce formulas for y_c and y_d by plugging in the form of x_c.

8.30 **(a)** Let $\check{h}[n] = h[-n]$ be a time-reversed version of h. Notice that $\check{h}$ is supported from $n = -31, \ldots, -18$. The n-th element of the convolution $y = x * h$ is just

$$y[n] = \sum_{\ell} x[\ell]h[n - \ell] \tag{6}$$

This is a dot product of y with a version of $\check{h}$ which has been shifted to the right by n samples:

$$y[n] = \left\langle x, \mathcal{D}_n \check{h} \right\rangle. \tag{7}$$

The shifted version $\mathcal{D}_n\check{h}$ is supported from $-31+n,\ldots,-18+n$. The smallest n for which this overlaps with the support $21,\ldots,31$ of x occurs when

$$-18+N_1=21, \tag{8}$$

while the largest n for which $\mathcal{D}_n\check{h}$ overlaps with x is

$$-31+N_2=31. \tag{9}$$

So, the output y is supported from $N_1=39$ to $N_2=62$.

(b) For $n=0,\ldots,31$, let

$$\bar{x}_1[n]=x_1[n+21 \bmod 32], \tag{10}$$

$$\bar{h}_1[n]=h_1[n+18 \bmod 32]. \tag{11}$$

Notice that hte linear convolution $\bar{x}_1 * \bar{h}_1[n]$ is simply $y[n+39]$. Moreover, because length$(\bar{x}_1)$ + length$(\bar{h}_1)-1\le 32$, linear convolution and cyclic convolution of these two signals are equivalent. So,

$$\mathrm{DFT}^{-1}\left\{\,\bar{H}_1[k]\bar{X}_1[k]\,\right\}[n]=y[n+39], \qquad n=0,1,\ldots,32. \tag{12}$$

By the cyclic shift property of the DFT,

$$\mathrm{Y}_1[k]=\bar{H}_1[k]\bar{X}_1[k]\exp\left(-j\frac{2\pi k\times 39}{32}\right). \tag{13}$$

giving

$$y_1[n]=y[(n-39 \mod 32)+39]. \tag{14}$$

(c) Both sequences are treated as length 32, we can set $N=32+32-1=63$. For any $N<63$, there will be spurious (incorrect) nonzero components in the output for $y[0],\ldots,y[63-N-1]$.